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ORIGINAL ARTICLE

Truth versus validity in mathematical proof

Viviane Durand-Guerrier

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Abstract In mathematics education, it is often said that mathematical statements are necessarily either true or false. It is also well known that this idea presents a great deal of difficulty for many students. Many authors as well as researchers in psychology and mathematics education emphasize the difference between common sense and mathematical logic. In this paper, we provide both epistemological and didactic arguments to reconsider this point of view, taking into account the distinction made in logic between truth and validity on one hand, and syntax and semantics on the other. In the first part, we provide epistemological arguments showing that a central concern for logicians working with a semantic approach has been finding an appropriate distance between common sense and their formal systems. In the second part, we turn from these epistemological considerations to a didactic analysis. Supported by empirical results, we argue for the relevance of the distinction and the relationship between truth and validity in mathematical proof for mathematics education.

Keywords Truth · Logical validity · Proof · Syntax · Semantics · Mathematics education · Didactics · Philosophy of logic and language · Epistemology

V. Durand-Guerrier
LEPS, Laboratoire d'étude du Phénomène Scientifique (Laboratory for Studying the Scientific Phenomenon), EA 4148, Université de Lyon; Université Lyon 1, 38 bd Niels Bohr, 69622 Villeurbanne Cedex, France

V. Durand-Guerrier (✉)
Institut Universitaire de Formation des maîtres (Teacher Training Institute), Université Lyon 1, 5 rue Anselme, 69004 Lyon, France
e-mail: vdurand@univ-lyon1.fr
URL: http://lirdhist.univ-lyon1.fr

1 Introduction

In mathematics education, it is often said that mathematical statements are necessarily either true or false. It is also well known that, for many students, it is difficult to deal with this conception. Many authors as well as researchers in psychology (Wason & Johnson-Laird 1977; Politzer 1991; Johnson-Laird 1986; Inglis & Simpson 2006) and researchers in mathematics education (Radford 1985; Arsac et al. 1992; Legrand 1993; Selden & Selden 1995; Dubinsky & Yiparaki 2000; Hanna 2000; Hoyles & Küchemann 2003; Rogalski & Rogalski, 2004) emphasize the difference between common sense and mathematical logic. In this paper, we provide both epistemological and didactic arguments to reconsider this point of view, taking into account the distinction made in logic between truth and validity on one hand, and syntax and semantics on the other. First, we will examine the contribution of some prominent logicians to the clarification of the distinction and the relationship between truth and validity, emphasizing the articulation between syntax and semantics. In the second part, we will use two examples to illustrate the relevance of these epistemological considerations for didactic analysis.

2 Epistemological considerations

The specific role of epistemology in mathematics education research is defended by many authors like Arsac (1987), Artigue (1991), Sierpinska & Lerman (1996), or Dorier (2000). Dorier (2000) proposes a rather vague sense of epistemology, linked to any consideration touching the evolution of knowledge. Here, in our consideration of reasoning and proof, we use epistemology in a more restricted sense. Thus, our goal is to understand in precisely what sense logic can be considered to be an epistemological reference for the didactic analysis of proof and reasoning in mathematics. This leads us to consider "La province de la logique," as analyzed by Engel (1989) in his consideration of philosophers from Aristotle to Quine, in particular Frege, Russell, Wittgenstein and Tarski. Reading these authors makes it clear that the need to determine the appropriate distance between common sense and logic is at the very core of their theoretical constructions. Aristotle, to start at the beginning, considered only propositions[1] to be either true or false, affirming that many other linguistic entities, such as prayers, orders, etc., are not. Both Frege and Russell gave a definition of implication that agrees with common sense when the antecedent is true but not when the antecedent is false, arguing that they needed to do so to build a coherent theory for valid inference, and Quine shared this point of view. Wittgenstein elaborated a semantic version of propositional calculus that simultaneously proclaims the autonomy of the system and its applicability to ordinary language, and Tarski constructed a semantic definition of a true statement as both "materially adequate" and "formally correct." The main categories that emerge from these investigations are syntax and semantics on the one hand, and truth and validity on the other. Syntax and semantics are clearly related to "form and contents" (Sinaceur, 1991, 2001), which is a crucial issue in mathematics education, while truth and validity are at the very core of the process of mathematical proof. Obviously, syntax and semantic are more general categories; however, as we shall see, syntactic and semantic methods concern both truth and validity.

2.1 Truth and validity in Aristotle' syllogism theory

It is well known that formal logic begins with Aristotle' syllogism theory as presented in his Prior Analytics (Aristote, 1992). As Lukasiewicz put it, for Aristotle, pure logic is what remains when material has been taken away (Lukasiewicz, 1951, 1972, p. 22). To build his system, Aristotle, in *On Interpretation* (Aristote, 1989), extracted formal statements from ordinary-language sentences; he gave a standard form to quantified statements and emphasized the distinction between contradiction (which oppose two necessarily different truth values, e.g., every A is B/some As are not Bs) and contrariety (a more radical opposition that offers the possibility of both statements being false, e.g., every A is B/no A is B). Then, in the *Prior Analytics*, Aristotle offers a precise definition of a syllogism: namely, a conditional statement with two premises and a conclusion (all quantified statements) that respect a set of precise constitutive rules, like, for example: "If some *A*s are *B* and every *B* is *C*, then some *A*s are *C*" (1), or "If every *A* is *B* and some *B*s are *C*, then some *A*s are *C*" (2). Following this definition, Aristotle then classifies these syllogisms into two categories: those that lead from truth to truth, whatever be the interpretation of the terms *A*, *B* and *C* as in (1); and those which might have true premises and a false conclusion in some interpretation as in (2). To do this, Aristotle first claims that some syllogisms are obviously valid, meaning that everybody would agree with the fact that they preserve truth. The most famous is the first syllogism of the first figure[2]: "If every *A* is *B* and every *B* is *C*, then every *A* is *C*." Then, he gives some conversion rules that preserve validity such as replacing "Some *A*s are *B*" by "Some *B*s are *A*," which are "obviously" equivalent (synonymous). He is then able to prove, by syntactic means, that some syllogisms are logically valid (are truth-preserving). Moreover, for every syllogism that can be constructed in his system, he is able either to prove syntactically that it is valid, or to give a counterexample showing it not to be valid. By proceeding in this manner, Aristotle is simultaneously using both syntactic and semantic processes. As a consequence, he emphasized the distinction between truth in an interpretation and logical validity. Aristotle qualifies a truth as "necessary" when it is the conclusion of a valid syllogism whose premises are true, and opposes it to a *de facto* truth, or a truth obtained merely as a consequence of another truth. Even if Aristotle's system is clearly not sufficient for the needs of mathematical reasoning (in particular, he did not explicitly introduce logical connectors like implication), he nevertheless developed fundamental logical concepts that remain essential in modern logic.[3] Thus, he is acknowledged as a precursor by many authors, who, like Largeault (1972), consider that Aristotle's use of both semantic interpretation and formal derivation attests of his "genial lucidity."

2.2 The revival of formal logic in the late nineteenth century

Although logic has always been at the center of philosophers' and scientists' inquiries, it experienced spectacular developments only at the very end of the nineteenth and the

[1] In logic, a proposition is a linguistic entity that is either true or false.

[2] A syllogism has two premises and a conclusion; each premise is a proposition with a subject term and a predicate term (an attribute); the middle term occurs twice in the premises. It does not occur in the conclusion. Its position determines four figures. For example, in the first figure, the middle term is once the predicate, once the subject; in the second figure, it is twice the predicate.

[3] A very clear presentation of Aristotle's Syllogistic can be found in Lukasiewicz (1951), who presents this text as an introduction to formal logic.

beginning of the twentieth century. Frege and Russell are two prominent figures in this revival.

The revolution introduced by Frege consisted of the elaboration of a symbolic language, enriched using mathematical language, to translate logical operations (connectors and quantifiers). In his paper (Frege 1882), "*Uber die Wissenschaftliche Berechtigung einer Begriffschrift*,"[4] Frege exposed his project of renewing logic to attain perfect rigor in mathematical reasoning. Such rigor could not be achieved, according to him, using ordinary language. Frege admitted that for ordinary contexts, our experience could be sufficient to avoid many errors, but that this is not the case for more complex domains, such as mathematics. Indeed, rules of logic expressed in ordinary language are insufficient to ensure that the chain of inference does not contain any gaps. This reasoning is supported by some historical incorrect proofs such as those of Cauchy, which ultimately led to the distinction between uniform convergence and convergence (Durand-Guerrier and Arsac, 2003, 2005). Frege's ideography is capable of revealing the deep structure of mathematical statements, by formalizing the distinction between singular and general statements and by showing the scope of this generality in relation to connectors like negation and implication. This is particularly important, as it allows one to resolve ambiguities that might be encountered in ordinary language. This is the case in French, for example, in statements with the logical form "All *A*s are not *B*." Such a statement might be understood as "it is not the case that all *A*s are *B*," or to put in a different way "there exist some *A*s that are not *B*" (1). However, it could also be understood as "None of *A*s is *B*" (2). Although in French, the right interpretation is given by (1), in some contexts, French-speakers can legitimately make the second interpretation. This semantic ambiguity is well known to linguists dealing with French language (Fuchs, 1996) and we find it also in mathematical classroom in Tunisia (Durand-Guerrier & Ben Kilani 2004) and with French teachers in mathematics. Thus, for example, the sentence "Toutes les boules ne sont pas rouges" (*All the balls are not red*), leads to two different interpretations "Il y a au moins une boule qui n'est pas rouge" (*At least one ball is not red*) (1), or "Aucune boule n'est rouge" (*No ball is red*) (2). Expressing the logical structure of the statement in predicate calculus requires choosing between the two interpretations, and thus disambiguating the signification. It is significant that a literal translation of the French sentence into predicate calculus provides the second interpretation, as the negation is within the scope of the universal quantifier. To get the first interpretation, it is necessary to extract the negation from the sentence and to put it at very beginning or to change the universal quantifier into an existential quantifier. Frege discusses these difficult questions in some detail, but they are rarely taken into account in the mathematics classroom, at least in France and in Tunisia. Even more significant difficulties arise when both implication and negation are involved in quantified statements. Selden & Selden (1995) show that undergraduate students in the United States experience many difficulties when asked to translate mathematical statements given in ordinary language into first-order predicate logic.

One important aspect of Frege's contribution is his definition of implication and negation (Frege 1918a, b, 1923, 1971). He defined the relationship of implication "if *B*, then *A*" as equivalent to "not (not *A* and *B*)," which is false only in the case where the antecedent is true and the consequent false. Frege rejected the objections to this definition by arguing not only for the need for logic to liberate itself from the use of ordinary language but also the necessity of establishing a certain distance from the ordinary use of language. Thus, his ideography presents itself as a purified version of ordinary language, an approach in continuity with the Aristotelian perspective. Nevertheless, unlike Aristotle, Frege incorporated logical connectors and did not use vernacular language at all. He also insisted on the fact that this definition of implication was necessary to be clear about the cases where a conditional statement is true and so to clarify the use of a true conditional statement in an inference, such as "*A*; and if *A*, then *B*; hence *B*." The definition given by Frege is similar to "material implication" as defined by Russell (1903), who insisted on the fact that, although it looks unnatural, there must be a logical relation between two propositions of the sort that either the antecedent is false or the consequent is true. Russell claimed that the natural concept of implication is a generalized conditional that asserts that every material implication (propositional conditional) of a certain class is true.[5] However, quantified logic is an extension of propositional logic, which is at the very heart of deductive reasoning. A greater difficulty arises from the fact that propositional logic is developed as an axiomatic system according to a syntactic perspective (logical theorems are derived from axioms by the application of inference rules in the system itself), while quantified logic as defined by Frege and Russell clearly needs a semantic perspective to take into account interpretation and issues concerning the domain of quantification. Wittgenstein, by elaborating a semantic version of propositional logic, opened up a new

[4] English translation "On the scientific justification of a conceptual notation" in Frege 1972, pp. 83–90.

[5] The question of knowing which is the right notion of implication is discussed in (Durand-Guerrier 2003).

path to a solution, which was ultimately realized by Tarski with his semantic definition of the concept of truth.

2.3 A semantic perspective on propositional logic

A disciple of Russell and Frege, Wittgenstein, proposed a formal system constructed using a semantic perspective in his first treatise, entitled *Tractatus logico philosophicus* (Wittgenstein, 1921, 1922).[6] His main purpose here was to formalize the notion of a proposition, i.e., a linguistic entity that is either true or false. The components of the system are "propositional variables," meaning symbols or variables that could be interpreted as propositions in some particular piece of discourse. Two principles govern the system. First, the principle of bivalence, proposing that there are exactly two truth values in the system and, second, the principle of extension, which asserts that the truth-value of a complex sentence is entirely determined by the truth values of its elementary components. Truth-tables are introduced to define all the possibilities of truth distribution in the overall system. For two variables, for example, there are sixteen possible combinations. According to Granger, 1990, p. 4, "these do not refer to any object of our thought, but evoke systems of possibility concerning the truth or falsity of the propositions that are thereby connected together." ("*elles ne renvoient à aucun objet de pensée, mais évoquent des systèmes de possibilités pour la vérité et la fausseté des propositions qu'elles connectent.*"). Of course, among these possibilities, we find all the classical logical connectors, such as conjunction, implication, equivalence, and disjunction. The formal propositions in the system are built up in a recursive manner from the propositional variables and the connectors so that it is possible to build their truth-tables, which indicates the truth-value of a complex proposition for each distribution based on the truth-value on its elementary components. Among these formal propositions, some take the truth-value "true" for every distribution: they are named tautology and play an essential role in the system: tautologies support deduction, while interpretation of formal propositions that are neither tautologies, nor contradictions[7] "*speak about facts in the world, to describe the state of things*," and consequently, it is not the role of logic to decide on the truth of the particular interpretations (this point was already emphasized by Frege). The main role of logic is to establish which inferences are valid. Wittgenstein shows how tautologies support deduction in the case of Modus Ponens saying that as we can show that "$((p \Rightarrow q) \wedge p) \Rightarrow q$" is a tautology, it is clear that "q" follows of "$(p \Rightarrow q) \wedge p$." The important novelty in this system is the fact that there is no use of a rule for the conclusion, as was the case in Frege's and Russell's systems. Indeed, the fact that a formal statement is or is not a tautology (or a contradiction) depends only on its structure. In the case of complex propositions, the truth-table method offers a decisive mean to determine this. As a consequence, proof in logic, and the logical proof of a mathematical statement are two different things. As most of the classical inference rules can be associated with a tautology, Wittgenstein's system may be considered as a "theory of valid inference." This was widely popularized by Quine, who claimed that this only was implication (Quine 1950). Thus, what emerges from the *Tractatus* is an autonomous formal system aimed at providing an adequate description of the *facts in the world and the state of things*. In this treatise on propositional calculus, Wittgenstein brilliantly overcame this tension between formalism and a description of the world,[8] but when it comes to quantified logic, this question was scarcely explored. Some years later, Tarski, in a famous paper first published in Polish in 1933 did for quantified logic what Wittgenstein had achieved for propositional logic.

2.4 A definition of truth materially adequate and formally correct

In his 1933 paper, entitled in English "*The concept of truth in languages of deductive sciences*," Tarski indicated that his purpose was to construct a definition of the expression "true proposition" that would be materially adequate and formally correct (Tarski 1933a, b, 1972, 1983, p. 159). Although Tarski seems not to have read Wittgenstein, it appears that he knew about the semantic perspective in propositional calculus. Tarski's project is clearly inscribed in a perspective of bridging formal systems and reality. He emphasized this point once again in a paper from 1944, where he refers clearly to the classical Aristotelian conception of truth that could be expressed in modern language under the following definition: "the truth of a proposition lies in its agreement (or correspondence) with reality; or a proposition is true if it designates an existent state of things (Tarski 1944a, b, 1974, pp. 270–271)."

To elaborate a recursive construction of truth for propositions, Tarski introduced the more general concept of "satisfaction of a propositional function (a predicate) by such or such objects," taking into account the fact that "complex propositions are not aggregates of propositions, but obtained from propositional functions" (Tarski 1933a, b, 1972, 1983, p. 193). This definition highlights the fact

[6] There are obviously many others philosophical matters raised in this treatise. Here, I propose a reading based on a purely didactic perspective (Durand-Guerrier 2006).

[7] A contradiction take the truth-value "false" for every distribution.

[8] This is developed in (Durand-Guerrier 2006).

that to state the truth of a propositional function, it is necessary to work in a given domain of reality in which there are existent states of things about which we are able to say something concerning their truth. For Tarski, a domain of reality might equally well be material reality, mathematical theory or a local theory elaborated *pour les besoins de la cause*,[9] as we will see in the next paragraph. It is then possible to construct recursively the criteria for the satisfaction of a complex formula of predicate calculus in any structure on a nonempty domain using an interpretation for each letter of the formula. Doing this, it is possible to define the notion of "model for a formula," which constitutes an interpretative structure (for example, a mathematical theory) in which the formula is satisfied by every relevant sequence of objects. This allows Tarski to define the fundamental notion of "logical consequence in a semantic point of view": a formula G follows logically from a formula F if and only if every model for F is a model for G (Tarski 1936c, 1972). This then means that the formula "$F \Rightarrow G$" is true for every interpretation of F and G in every nonempty interpretative structure (Quine 1950). For example, in this semantic context, "$Q(x)$" is a logical consequence of "$P(x) \wedge (P(x) \Rightarrow Q(x))$". Note that this is an extension of the corresponding result by Wittgenstein, in the sense that "$Q(x)$," and "$P(x) \wedge (P(x) \Rightarrow Q(x))$" are not propositional variables, but propositional functions (predicates), so that it is not possible to use the truth-tables directly [in predicate calculus, only closed formulae (formulae without free variables) can be considered as propositional variables].

A clear presentation of this notion of logical consequence from a semantic perspective can be found in (Quine 1950). This author, in the continuity of his predecessors, developed logical tools that allowed the formalization of propositions that remained as close as possible to ordinary language and natural modes of reasoning, although without hesitating to establish a distance from it if necessary. He thereby succeeded in providing valuable tools for the formal analysis of language, reference and inference.

2.5 Semantics and syntax: a model theoretic point of view

The model-theoretic approach was developed by Tarski in his 1936 book *Introduction to logic and to the methodology of the deductive sciences* (Tarski 1936a, b, 1969). He named his method the "Methodology of the deductive sciences," and then presented it using an example (the congruence of segments). Given a deductive theory, it is possible to consider an axiomatic system as a formal language (without any defined objects), and then to reinterpret the system using other interpretations. Such interpretations in which the axioms are true are named models of the axiomatic system. Of course, the initial theory is among the models of the system. Tarski (1936a, b, 1969) established the following important results:

> "Every theorem of a given deductive theory is satisfied by any model of the axiomatic system of this theory; moreover at every theorem one can associate a general logical statement logically provable that establishes that the considered theorem is satisfied in any model of this type (…)." (Deduction theorem).

> "All the theorems proved from a given axiomatic system remain valid for any interpretation of the system."

These two fundamental theorems illustrate the relationship between semantics and syntax and lead to a very important method of proving that a statement is not a logical consequence of the axiom of a theory. This kind of proof, called "proof by interpretation," consists of providing a model of the theory that is not a model of the formula associated with the statement in question.

Doing this, Tarski clarifies the distinction between truth in an interpretation, and truth as a logical consequence of an axiomatic system, which recovers Aristotle's original distinction between necessary and *de facto* truth.[10] As Sinaceur (1991) has demonstrated, this leads to many results in advanced mathematics. However, this method is also fruitful in more elementary domains as we will show now, to illustrate this methodology. The example is taken from the famous international mathematical competition for students called the *Math Kangaroo* (in France, *le Kangourou des Mathématiques*).[11] The item that we present here is from the French 1994 competition for eighth-grade students (14 years old). Below is the text of the last item in the competition:

Les gens malins répondront tous juste à cette question;
All clever people will give the right answer to this question;
Ceux qui répondent au hasard ne sont pas malins.
Those who answer by guessing are not clever.
Alors il est certain que :
Then it is certain that:
A. *Tous ceux qui répondent au hasard répondent faux.*
All those who answer by guessing will give a wrong answer.
B. *Ceux qui sont malins répondent au hasard.*
Those who are clever will answer by guessing.

[9] In French, this expression means that you do something in order to solve a specific problem.

[10] see Sect. 2.1.

[11] http://www.mathkang.org/concours/kangsansf.html.

Table a continued

C. *Ceux qui répondent juste sont malins.*
Those who give the right answer are clever.
D. *Ceux qui s'abstiennent à cette question sont malins.*
Those who don't answer this question are clever.
E. *Les réponses A, B, C et D sont fausses.*
The answers A, B, C and D are false.

To answer the question, students had to choose one of the five responses. Reading the answer proposed by the authors—the correct answer is E—raises some doubts. Are we really sure that the sentences A, B, C and D are false? The question is how do you determine the truth-values of the sentence A, B, C and D? A priori, the questions of certainty are related to questions of necessity, and even, in this case, of logical validity. To discuss this, we will use Tarski's methodology of the deductive sciences, as the structure of this item is appropriate for constructing a mini deductive theory. Indeed, we have a domain of discourse: *the population of all those who answer the Kangaroo question in 1994 in France*, and two axioms: *all clever people will give the right answer to this question* (A_1); *those who answer by guessing are not clever* (A_2). Moreover, the question refers predominantly to semantic issues. First, it concerns *objective certainty* ("It is certain that") rather than *subjective certainty* ("I am certain that"), so that the answer is expected to be formulated within the same framework as the question. Within this framework, however, there are certain ambiguous references, because, on one hand, the one who answers contributes to the definition of the domain of discourse, and on the other hand the fifth sentence says something about the truth of the four other sentences.

To construct an axiomatic system in predicate calculus, which we will name T, we need to define relevant predicates. We will associate the following states; "être malin (*to be clever*)," "répondre juste (*to answer correctly*)," "répondre faux (*to answer incorrectly*)," "s'abstenir (*not to answer*)," "répondre au hasard (*to answer by guessing*)," with the letters $\boldsymbol{m}, \boldsymbol{j}, \boldsymbol{f}, \boldsymbol{s}, \boldsymbol{h}$, respectively.

As far as the axioms are concerned, we have already mentioned two: *all clever people will give the right answer to this question* (A_1); *those who answer by guessing are not clever* (A_2). These may be formalized by "$\forall x\ (\boldsymbol{m}(x) \Rightarrow \boldsymbol{j}(x))$" ($A_1$) and "$\forall x\ (\boldsymbol{h}(x) \Rightarrow \neg\boldsymbol{m}(x))$" ($A_2$), but other axioms are also needed. Indeed, as is generally the case when formalizing, it is necessary to express conditions that remain implicit in cases of informal reasoning. In this situation, it is necessary to state that "*All those who answer the question either do not give any answer or give the right answer or give the wrong answer* (*exclusive*)," which corresponds to four new axioms: $\forall x\ (\boldsymbol{s}(x) \vee \boldsymbol{j}(x) \vee \boldsymbol{f}(x))$ (A_3); $\forall x\ \neg(\boldsymbol{j}(x) \wedge \boldsymbol{f}(x))$ (A_4); $\forall x\ \neg(\boldsymbol{j}(x) \wedge \boldsymbol{s}(x))$ (A_5); $\forall x\ \neg(\boldsymbol{f}(x) \wedge \boldsymbol{s}(x))$ (A_6). We have chosen to complete our system with the assertion that "*There exists at least one person in each category*," giving five new axioms: $\exists x\, \boldsymbol{j}(x)$ (A_7); $\exists x\ \boldsymbol{f}(x)$ (A_8); $\exists x\ \boldsymbol{s}(x)$ (A_9); $\exists x\ \boldsymbol{m}(x)$ (A_{10}); $\exists x\ \boldsymbol{h}(x)$ (A_{11}).

Those who answer the question have to decide for each of the four statements, if it is certain that they are true, or if it is certain that they are false. However, it might occur that it is not possible to decide in precisely these terms, in other words, that it is neither certain that the statement is true nor certain that it is false. Indeed, when the student responds to the question, there will be a realization of the axiomatic system, but before the response is provided, the realization has not been determined, and so cannot be known. In other words, to be certain of the truth-value of one sentence out of A, B, C and D, it is necessary that these truth-values are independent of the responses, which contribute to the realization of the system. This can be the case only if either the formula corresponding to the sentence or its negation is a consequence of the axioms. Here, we will just treat the cases of the two responses A and B. Sentence A "*All those who answer by guessing give a wrong answer*" is formalized by the formula Φ: $\forall x\ (\boldsymbol{h}(x) \Rightarrow \boldsymbol{f}(x))$, and sentence B "*Those who are clever answer by guessing*" is formalized by the formula Ψ: $\forall x\ (\boldsymbol{m}(x) \Rightarrow \boldsymbol{h}(x))$. A rather natural conjecture is that, both sentences are false, although some difference is perceptible between the two sentences. Sentence A might be true, while sentence B seems to be necessarily false. First, we prove that *B* is necessarily false, which means that *not B* follows logically from the axioms. For this, we provide a proof that "$\neg\Psi$ is a logical consequence of T":

Step 1: $\neg\Psi$ is logically equivalent to $\exists x\ (\boldsymbol{m}(x) \wedge \neg \boldsymbol{h}(x))$; *Step 2*: the axiom A_1 is logically equivalent to $\forall x\ (\boldsymbol{m}(x) \Rightarrow \neg\boldsymbol{h}(x))$; *Step 3*: we use Copi's method of natural deduction)[12]:

(1)	$\forall x\ (\boldsymbol{m}(x) \Rightarrow \neg\boldsymbol{h}(x))$	Premise
(2)	$\exists x\ \boldsymbol{m}(x)$	Premise
(3)	$\boldsymbol{m}(\omega)$	Existential instantiation on (2)
(4)	$\boldsymbol{m}(\omega) \Rightarrow \neg\boldsymbol{h}(\omega)$	Universal instantiation on (1)
(5)	$\neg\boldsymbol{h}(\omega)$	Modus Ponens on (1) and (3)
(6)	$\boldsymbol{m}(\omega) \wedge \neg\boldsymbol{h}(\omega)$	Conjunction on (3) and (5)
(7)	$\exists x\ (m(x) \wedge \neg\boldsymbol{h}(x))$	Existential Generalization

[12] The method of Copi (1954) is briefly described in the Appendix.

This proves that $\neg\Psi$ is a logical consequence of **T**. As **T** is consistent (**T** has at least one model), any model of **T** is a model of $\neg\Psi$; hence, it is certain that sentence B is false.

Now we can prove the conjecture concerning sentence A by way of a proof by interpretation, providing two models of the axiomatic system: a model Σ_1 of **T**, which is a model of Φ, and a model Σ_2 of **T**, which is not a model of Φ. It is clear that a model of **T** needs to have at least three elements. Taking the domain $K = \{\alpha, \beta, \delta\}$, we can define the model of **T** by the extension of the five predicates, so that the interpretation of the axioms in the model are true; we note X* the extension of the property X that interprets *x*. For Σ_1, we suppose that: $J^* = \{\alpha\}$, $F^* = \{\beta\}$, $S^* = \{\delta\}$, $M^* = J^* = \{\alpha\}$, $H_1{}^* = F^* = \{\beta\}$, while for Σ_2, we suppose that: $J^* = \{\alpha\}$, $F^* = \{\beta\}$, $S^* = \{\delta\}$, $M^* = J^* = \{\alpha\}$, $H_2{}^* = S^* = \{\delta\}$. In Σ_1, the interpretation of Φ is true, while in Σ_2 it is false. Indeed, as we have $H_2(\delta)$ and $\neg F(\delta)$, δ is a counterexample to the general conditional in Φ. Thus, neither Φ nor $\neg\Phi$ is a logical consequence of **T**. As a consequence, those who answer the test cannot be certain that sentence A is false. However, if they give the answer A, they are wrong. Note that, in French, it is possible to use the term "*faux*" either as an adjective or as an adverb, meaning that the text of the question has two possible meanings: "la phrase A est fausse" (sentence A is false), or "répondre A est faux" (it is wrong to answer A). There is a second ambiguity in French, as the term "réponse" could designate the sentence itself or the act of answering. Thus, the fifth question is also ambiguous. It can be understood as "It is certain that sentences A, B, C and D are false"; we have seen that it is not the case, so sentence E is false, and consequently, none of the sentence is true, but it could also be understood as "It is certain that answering A, B, C or D is wrong," in that case, E is true and so answering E is right.

This example illustrates the large scope for applying the methodology developed by Tarski. It also shows a path for teaching it to undergraduate students, to give a concrete signification to fundamental logical-mathematical concepts such as *truth*, *satisfaction*, *validity*, *axiom*, *model of a theory*, which are defined extrinsically for a formal system. This contrasts with Proof theory, which is concerned only with the study of the intrinsic property of syntactic deductibility, and so this example makes explicit the interplay between semantic and syntactic points of view, which is so essential to mathematical activity.

Arriving at the end of this first part, we hope we have managed, by means of epistemological insights, to illuminate the interplay between syntax and semantics on the one hand, and truth and validity on the other. These distinctions are clearly described and formalized by Tarski, and nowadays incorporated in the modern predicate logic. The second part of this paper will be devoted to illustrating the thesis that these epistemological considerations are relevant for mathematics education, especially concerning proof. Thus, these reflections help both in interpreting students' mathematical activity, and in dealing with rigor in advanced mathematical studies.

3 Application to didactic analysis

In this part, we aim to show that the previous epistemological considerations have some important applications in mathematics education. First, we will give evidence that considering mathematics as an activity leads to shifting the focus from general statements, which are the very end of the process, to singular statements, open statements, and hence contingent statements, dealing with mathematical objects, properties and relationships, which are involved throughout the process. From this perspective, it is clear that a strictly syntactic perspective is insufficient, both for the analysis of students' activity in constructing proofs, and for analyzing their written proofs. We will then show that a semantic point of view is relevant for analyzing difficulties in constructing proofs involving several quantifiers. In particular, this approach is useful for analyzing proofs in which statements of type "For all *x*, there exists *y*, such that $P(x, y)$" are used, or are to be proved (Durand-Guerrier and Arsac 2003, 2005).

3.1 A semantic point of view on students activity

In this section, we will try to make clear, by means of an example, what we mean by the duality between "working with mathematical objects and their properties" and "working with general statements."[13]

The example is drawn from (Arsac et al. 1992), which consists of an analysis of mathematical situations for teaching deductive reasoning to students of 12–13 years old. The situation is dedicated to the two rules: "an example that satisfies that a statement is not sufficient to conclude that this statement is true," and "a counterexample is sufficient to prove that a statement is false." The problem submitted to the pupils is to know if the sentence "for every n, $n^2 - n + 11$ is a prime number" is true or not. Pupils first work alone, then in small groups; each group writes a poster; the posters are then collectively commented upon and there is a debate about the correctness of the answers. Students must decide on the truth-value of a general statement whose domain of quantification is infinite. In their presentation of the situation, the

[13] This example is developed in (Durand-Guerrier 2005, pp. 128–139). Notice that we use "general statements," where Jahnke (2008) use "closed general statements."

authors included a fragment of dialog between pupils concerning the truth-value of the sentence. The students are commenting a claiming that the sentence is true; there are two examples and a confusion between odd and prime. Other pupils have found the obvious counterexample *11*; so they argue that, as the sentence is not always true, it is false, which is the answer expected by the teacher. However, some pupils, like Marie, do not want to declare that the sentence is false:

(40) Elève: Y a une exception, donc c'est pas toujours.
Student: there is an exception; hence it is not always.
(41) Marie: Ça a été reconnu. À part ça, c'est toujours un nombre premier. Si on éliminait *11* ben…
Marie: That has been established. Except for this, it is always a prime number. What if we eliminated 11…? (…)
(63) Marie: Oui mais *22*, c'est le double de *11*, on peut peut-être essayer *33*, à mon avis ce sera aussi une exception.
Marie: Yes, but 22 is twice 11; we can maybe try 33; I think that this will also be an exception.

Finally, Marie accepts the claim that the sentence is false when another student provides the counterexample *25*,

(64) Marie: je crois qu'ils ont gagné, parce qu'il y a aussi *25* comme exception
Marie: I think they have won, because 25 is also an exception.

The debate went on about how many exceptions are needed to be convinced.

(76) Marie: Ça devient plus des exceptions parce que 22, 33 c'est tous des multiples.
Marie: They are no longer exceptions because 22, 33, are all multiples.

This shows that for Marie, the argument is not the number of counterexamples, but their relationship to the situation. She adds that to be sure of getting a true sentence, it is necessary to be under *100*. The authors report that asking the question anew some days later, several pupils declare the statement is false, citing the two counterexamples *11* and *25.*

Our interpretation is that those students who do not want to declare that the statement is false as soon as a counterexample is found are not considering the closed statement. They are working with the open statement "$n^2 - n + 11$ is a prime number," in which they substitute numerical values for n. First, as the numbers from one to ten satisfy this relationship, it is likely that many students consider the statement to be true. In this case, the "miraculous" counterexample *11* is not sufficient to change their assessment of the truth-value. There are numbers for which the statement obtained by substitution is true, and others for which the statement is false. This interpretation renders the answer given by "Geraldine" coherent, as she concludes that it is both true and false. The group of discussions reveals a disagreement between those students who consider the general statement and insist on the fact that "it is not always true, so it is false," and those students who remain focussed on the particular cases they have used to make up their minds. According to Jahnke (2008), we could also say that some students consider an *open general statement*, while others consider *a closed general statement.* At the end of the sequence, the teacher concludes that in mathematics a statement is either true or false, and that a counterexample is sufficient to prove that it is false, without, however, adding that this concerns strictly *closed general statements*. The implicit message is that in mathematics only closed general statements are ever involved. In this situation, integers are familiar objects that constitute the domain of reality in which the mathematical activity of students occurs. This activity may lead students to make various true assertions: *the sentence is false*; *the sentence is not always true*; *the sentence is true for all integers from one to ten*; *the sentence is sometimes true, sometimes false*; *the sentence might be true and false*; *the sentence is true except for 11*; *the sentence is true except for the multiples of 11*; *the sentence is false for every multiple of 11*; *it is impossible to determine all the numbers for which the sentence is true* (*or false*)…. We may suspect that asserting one or another of these sentences depends partly on the actions of the students during the phase of exploration of the problem. Of course, finding that many numbers satisfy the sentence may incline to try to save the statement by eliminating the counterexamples—which is a rational posture —, while seeing immediately that *11* is a counterexample may focus on the falsehood of the statement, and may lead the student to stop the exploration, which is a priori not so rational, because it might be that in other situations the obvious counterexample is the only one! Of course, the equivalence between "there exists x such that not $F(x)$" and "not for all x, $F(x)$" is logically valid; however, a rigid application of this rule, independently of the mathematical context, meaning from a syntactic point of view, is not very productive in terms of developing mathematical abilities. In this perspective, it would be relevant to change this kind of situation, proposing open sentences, and asking for the largest domain for which the sentence is true. In this case, pupils cannot give a definitive answer, because they cannot characterize the examples and counterexamples. Nevertheless, it would allow the teacher to take into account the various possible postures of students mentioned above. Moreover, in other cases, such a question may lead to the elaboration of one or more relevant theorems.

3.2 Difficulties with multiquantified statements

Many undergraduate students face serious difficulties when studying calculus, especially when they have to deal with multiquantified statements (Dubinsky and Yiparaki 2000; Selden & Selden 1995; Chellougui 2003). A wide spectrum of difficulties concerns the case of statements of the type "for all x, there exists y, such that $F(x, y)$" where F is a binary relation, so-called AE statements following Dubinsky's nomenclature. In Durand-Guerrier and Arsac 2005, we develop arguments based on empirical observations showing that these difficulties are closely related to a specific reasoning rule, that we call "the dependence rule," connected with the repeated use of an AE statement. The following incorrect proof will illustrate this point.[14] We first recall a well-known theorem.

Theorem 1 (mean-value theorem) *Let us consider two real numbers a and b such that $a < b$ and a real function f defined on the closed interval $[a;b]$. If f is continuous on $[a;b]$ and differentiable on the open interval $]a;b[$, then there is a point c in the open interval such that $f(b) - f(a) = (b - a)f'(c)$, where $f'(c)$ is the first derivative of the function f at c.*

The reader should note that this statement is actually an AE statement, although the universal quantification on f is implicit. The same remark applies to the theorem to be proven, which is a generalization of the previous one applied to two functions.

Theorem 2 (Cauchy's mean-value theorem) *Let us consider two real numbers a and b where $a < b$ and two real functions f and g defined on the closed interval $[a;b]$. If f and g are continuous on $[a;b]$, and differentiable on $]a;b[$, and if the first derivative g' of g is never equal to zero on $]a;b[$, then there is a real number c in $]a;b[$, such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$).*

A proof often provided by first-year science students consists of the following deduction of Theorem 2 from Theorem 1:

> Function f satisfies the conditions for applying Theorem 1; hence, there is a number c in $]a;b[$, such that$f'(c)(b-a) = f(b) - f(a)$. Also, g satisfies the conditions for applying Theorem 1; hence, there is a number c in $]a;b[$, such that $g'(c)(b-a) = g(b) - g(a)$. As g' is never equal to zero on $]a;b[$, then $g'(c) \neq 0$; hence, $g(b) - g(a) \neq 0$. The result follows from the quotient of the above two equalities.

This proof is invalid; it is possible to prove it by considering two functions for which it is not possible to choose the same number c. Note that the existence of counter examples is not completely obvious. Indeed, considering two polynomials with powers strictly under three, it is always possible to choose the same number c.[15] Nevertheless, the functions x^2 and x^3 on interval $[0; 1]$ provide a counterexample.[16]

We can find numerous other examples relating to the same mistake that we could term "forgetting the dependence." However, this faulty argumentation may also be interpreted as the application of an invalid rule of logic: "for all x, there exists y, such that $F(x, y)$," and "for all x, there exists y, such that $G(x, y)$"; hence, "for all x, there exists y such that $F(x, y)$ and $G(x, y)$ (R). This rule can be represented by the following formula in predicate calculus: $[(\forall x \exists y\ F(x, y)) \wedge (\forall x \exists y\ G(x, y))] \Rightarrow [\forall x \exists y\ (F(x, y) \wedge G(x, y))]$. The previous example of the Cauchy mean value theorem provides a structure, in which the interpretation of this formula is false, that proves that it is not logically valid.

A common mathematical practice to avoid this difficulty is to put indices by the letters following the existential quantifier to distinguish the two applications of the statement, but this practice, used by mathematicians, has no theoretical status, as bound variables as well as dummy variables can be changed without changing the meaning of the sentence. We have shown in Durand-Guerrier & Arsac 2005 that even expert mathematicians can fail to deal effectively with this difficulty.

Although the Cauchy mean value theorem is correct and can easily be proved using an auxiliary function, it is obvious that no mathematician or mathematics teacher will accept this proof as a correct one, but what about the following proof, encountered in a French textbook (Houzel, 1996, p. 27) addressed to first-year university students? The theorem to prove is a classic one:

"Given two functions f and g defined in a subset A of the set of real number, and a an adherent element of A, if $f(t)$ and $g(t)$ have h and k, respectively, for limits as t tends to a remaining in A, then $f + g$ has $h + k$ for a limit in a."

The proof proposed by the author is the following one[17]:

[14] This example is developed in French in (Durand-Guerrier and Arsac 2003); in English in (Durand-Guerrier and Arsac 2005) and in (Durand-Guerrier 2004).

[15] For a polynomial with power one, the derivative is a constant; for a polynomial with power two, we have $2c = b + a$.

[16] For functions x^2 and x^3 on the interval [0;1], the number c must be $\frac{1}{2}$ and $\sqrt{\frac{1}{3}}$, respectively.

[17] Our translation.

"By hypothesis, for all $\varepsilon > 0$, there exists $\eta > 0$ such that $t \in A$ and $|t - a| \leq \eta$ imply $|f(t) - h| \leq \varepsilon$ and $|g(t) - k| \leq \varepsilon$; thus, we have
$|f(t) + g(t) - (h + k)| = |f(t) - h + g(t) - k| \leq$
$|f(t) - h| + |g(t) - k| \leq 2\varepsilon$"

The proof is actually elliptic, and for a novice, the first assertion could be interpreted as the result of the application of Rule R, which means:

Given that

"for all $\varepsilon > 0$, there exists $\eta > 0$, such that $t \in A$ and $|t - a| \leq \eta$ imply $|f(t) - h| \leq \varepsilon$" (1) and
"for all $\varepsilon > 0$, there exists $\eta > 0$, such that $t \in A$ and $|t - a| \leq \eta$ imply $|g(t) - k| \leq \varepsilon$" (2),
we deduce:
"for all $\varepsilon > 0$, there exists $\eta > 0$, such that $t \in A$ and $|t - a| \leq \eta$ imply $|f(t) - h| \leq \varepsilon$ and $|g(t) - k| \leq \varepsilon$." (3)

As was the case for the Cauchy mean value theorem, sentence (3) is true, but it is not a logical consequence of premises (1) and (2); indeed, as we have shown, rule R is not valid, but in contrast with the incorrect proof, it is likely that some mathematicians (but not all) may consider this second proof to be correct, for it is always possible to find a number that holds for the two functions, while, as we said above, it is not always possible in the first case. The incorrect use of rule R can be found in many situations, even in situations where it leads students to "prove" a false statement. This illustrates a very important difference between an expert and a novice in mathematics. An expert in a mathematical field knows when it is *dangerous* to relax the rigorous application of rules of inference, while novices have to learn this at the same time as they acquire the relevant mathematical knowledge. These two aspects of mathematics cannot be learned separately.

4 Conclusion

Our epistemological considerations show clearly how methods of logic that simultaneously take into account semantic and syntactic point of view remain close both to natural reasoning and to mathematical reasoning and proof. These considerations open up the possibility of reconsidering the widely held view that there is a difference in nature between natural and mathematical reasoning. The first didactic example shows that to analyze a student's activity, it is relevant to adopt a semantic point of view that provides tools for interpreting the student's answers in a more positive manner. Thus, the semantic point of view brings us closer to a scientific approach than a strictly syntactic point of view. The second didactic example shows that in advanced mathematics, logic and mathematic are closely intertwined, inviting teachers to make more rigorous use of the tools supplied by logic in their teaching. Throughout the paper, we have focused on the importance of the distinction and the relationship between truth and validity, which is one of the central issues studied in logic and analytic philosophy, but which is widely neglected in mathematics education. Although, as Glaeser (1973) claims, quantified logic might be difficult to master, we would instead argue that the benefits it can bring to mathematics education make this approach worthwhile.[18]

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Appendix

Natural deduction in predicate calculus

Natural deduction systems provide a theoretical framework that reflects quite well the way that mathematicians reason. These systems provide rules for the elimination and introduction of connectors and quantifiers. The first such system was due to Gentzen (1935, 1955), but it has since been modified by both Quine (1950) and Copi (1954). According to Prawitz (1965) "Because of their close correspondence to procedures common in informal reasoning, systems of natural deduction have often been used in textbooks for pedagogical purposes." (ibid. p. 103). Besides classical rules for the introduction and elimination of propositional connectors, we find four rules for the elimination and introduction of quantifiers in one-place predicate formulae, accompanied by two restriction rules (Copi 1954, 2nd edition, 1965, pp. 79–83). In the way that logicians generally do, Copi uses a horizontal line between two statements to indicate a deduction (see Fig. 1). Besides these four rules and their two restrictions, in case of two-place (or more) predicate, it is necessary to introduce a *third restriction rule*: U.G. can be applied provided that *fa* contains no individual symbol introduced by E.I. (Copi 1954, 2nd edition, 1965, p. 112). As a consequence, if *w* has been introduced by applying E.I. after *a* was introduced by applying U.I., then U.G. cannot be applied to *faw*; it is necessary first to apply E.G. By combining the introduction and elimination of connectors and quantifiers, Copi's system provides rules that on one hand allow local control of validity by analyzing deduction step by step, and on the other hand, indicate, by paying attention to change in logical status for letters, when global control of validity is required.

[18] On this question, see also Epp (2003).

1. U.I. Universal Instantiation

$$\frac{(x)fx}{fa}$$

(x) expresses a universal quantification corresponding to "$\forall x$"

a is an individual constant and fa results from fx by replacing all free occurrences of x in fx by a.

2.U.G. Universal Generalization

$$\frac{fa}{(x)fx}$$

First restriction rule: *a denotes any arbitrarily selected element, without any assumption other than its belonging to the considered domain.*

3. E.G. Existential Generalization

$$\frac{fa}{\exists xfx}$$

a is any individual symbol.

4. E.I. Existential Instantiation

$$\frac{\exists xfx}{fw}$$

Second restriction rule: It is necessary to be aware of the interpretation of w: w is "any individual constant which has had no prior occurrence in that context and is used to denote the individual, or one of the individuals, whose existence has been asserted by the existential quantification."

Fig. 1 The four rules for the introduction and elimination of quantifiers (according to Copi 1954, 2nd edition, 1965, pp. 80–82)

A more detailed presentation of this framework and an example of the way we use it are developed in (Durand-Guerrier 2005, pp. 163–168).

References

Aristote. (1989). Organon: I. Catégories-II De l'interprétation, Traduction nouvelle et notes par Jean Tricot, Librairie philosophique J. Vrin.

Aristote. (1992) Organon: III. Premiers analytiques, traduction Jean Tricot, Librairie philosophique J. Vrin.

Arsac, G. (1987). L'origine de la démonstration: essai d'épistémologie didactique. *Recherches en Didactique des Mathématiques, 8/3*, 267–312.

Arsac, G., Chapiron, G., Colonna, A., Germain, G., Guichard, Y., & Mante, M. (1992). *Initiation au raisonnement déductif au collège*. Presses Universitaires de Lyon: I.R.E.M. de Lyon.

Artigue, M. (1991) Epistémologie et Didactique. *Recherches en Didactique des Mathématiques, 10/2.3*, 241–285.

Chellougui, F. (2003). Approche didactique de la quantification dans la classe de mathématiques dans l'enseignement tunisien. *Petit X, 61*, 11–34.

Copi, I. (1954). *Symbolic logic* (*2nd edition, 1965*). New York: The Macmillan Company.

Dorier, J. L. (2000) *Recherches en Histoire et en Didactique des Mathématiques sur l'algèbre linéaire. Perspective théorique sur leurs interactions*. les cahiers du laboratoire Leibniz no. 12, http://www.leibniz-imag.frLesCahiers.

Dubinsky, E., & Yiparaki, O. (2000). On students understanding of AE and EA quantification. *Research in Collegiate Mathematics Education IV. CBMS Issues in Mathematics Education, 8*, 239–289. American Mathematical Society, Providence.

Durand-Guerrier, V. (2006). Lire le Tractatus dans une perspective didactique, in Ouelbani, M. (dir.) *Thèmes de philosophie analytique*, Université de Tunis, faculté des Sciences humaines et sociales

Durand-Guerrier, V. (2005). *Recherches sur l'articulation entre la logique et le raisonnement mathématique dans une perspective didactique. Un cas exemplaire de l'interaction entre analyses épistémologique et didactique. Apports de la théorie élémentaire des modèles pour une analyse didactique du raisonnement mathématique*, Note de synthèse pour l'habilitation à Diriger les Recherches (HDR), Université Lyon 1, I.R.E.M. de Lyon

Durand-Guerrier, V. (2004) Logic and mathematical reasoning from a didactical point of view. A model-theoretic approach. In Electronic proceedings of the third conference of the European Society for Research in Mathematics Education, February 28–March 3, 2003, Bellaria, Italia.

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics, 53*, 5–34.

Durand-Guerrier, V., & Arsac, G. (2003). Méthodes de raisonnement et leurs modélisations logiques? *Le cas de l'analyse. Quelles implications didactiques. Recherches en Didactique des Mathématiques, 23/3*, 295–342.

Durand-Guerrier, V., & Arsac, G. (2005). An epistemological and didactic study of a specific calculus reasoning rule. *Educational Studies in Mathematics, 60/2*, 149–172.

Durand-Guerrier, V., & Ben Kilani, I. (2004). Négation grammaticale versus négation logique dans l'apprentissage des mathématiques. *Exemple dans l'enseignement secondaire Tunisien. Les Cahiers du Français Contemporain, 9*, 29–55.

Engel, P. (1989). *La norme du vrai Philosophie de la logique*. Paris: Gallimard.

Epp, S. (2003). The role of logic in teaching proof. *American Mathematical Monthly* (110)10, December 2003, pp. 886–899.

Frege, G. (1882). Über die wissenschaftliche Berechtigung einer Begriffsschrift. *Zeitschrift für Philosophie und Philosophische Kritik, 81*, 48–56. French translation in Frege (1971), pp. 63–69.

Frege, G. (1918a). Der Gedanke. Eine logische Untersuchung. *Beiträge zur Philosophie des Deutschen Idealismus, 1*, 58–77. French translation in Frege (1971).

Frege, G. (1918b). Die verneinung. *Eine logische Untersuchung. Beiträge zur Philosophie des Deutschen Idealismus, 1*, 143–157. French translation in Frege (1971).

Frege, G. (1923). Logische Untersuchungen. *Beiträge zur Philosophie des Deutschen Idealismus, 3*, 36–51. French translation in Frege (1971); English translation in Frege (1972), pp. 83–90.

Frege, G. (1971). *Ecrits logiques et philosophiques*. Paris: Le Seuil.

Frege, G. (1972). *Conceptual notation and related articles*. Oxford: Clarendon Press.

Fuchs, C. (1996). *Les ambiguïtés du français*, Collection l'essentiel français Ophrys.

Gentzen, G. (1935). Untersuchungen über das logische Schliessen. *Math Zeitschr, 39*, 176–210, 405–431. French translation in Gentzen (1995)

Gentzen, G. (1955) *Recherches sur la déduction logique* (translated by R. Feys and J. Ladriere). Paris: PUF.

Glaeser, G. (1973). *Mathématiques pour l'élève professeur*. Paris: Hermann.

Granger, G. G. (1990). *Invitation à la lecture de Wittgenstein*. Aix en Provence: Alinea.

Hanna, G. (2000). Proof, explanation and exploration, an overview. *Educational Studies in Mathematics, 44*, 5–23.

Houzel, C. (1996). *Analyse mathématique. Cours et exercices*. Paris: Belin.

Hoyles, C., & Küchemann, D. (2003). Students'understandings of logical implication. *Educational Studies in Mathematics, 51*(3/2), 193–223.

Inglis, M., & Simpson, A. (2006). The role of mathematical context in evaluating conditionnal statements. In *Proceedings of the 30th conference of international group for the psychology of mathematics education, Vol. 3* (pp. 337–344), Prague, 17–21 July 2006.

Jahnke, N. (2008). Theorems that admit exceptions, including a remark on Toulmin. *ZDM—The International Journal on Mathematics Education*, this issue.

Johnson-Laird, P. N. (1986). Reasoning without logic. In T. Meyers, K. Brown & B. McGonigle (Eds.), *Reasoning and discourse processes* (pp. 14–49). London: Academic Press.

Largeault, J. (1972). *Logique mathématique, textes*. Paris: Armand Colin.

Legrand, M. (1993). Débat scientifique en cours de mathématique et spécificité de l'analyse. *Repères* no. 10, pp. 123–158

Lukasiewicz, J. (1951). *Aristotle's syllogistic from the standpoint of modern formal logic*. Oxford: Oxford University Press.

Lukasiewicz, J. (1972). La sylogistique d'Aristote (translated by F. Zaslaswski). Paris: Armand Colin.

Politzer, G. (1991). L'informativité des énoncés: contraintes sur le jugement et le raisonnement. *Intellectica, 11*, 111–147.

Prawitz, D. (1965). *Natural deduction, a proof theoretical study*. Stockholm: Almqvist and Wiksell.

Quine, W. V. O. (1950). *Methods of logic*. New York: Holy, Rinehart & Winston.

Radford, L. (1985) *Interprétations d'énoncés implicatifs et traitement logiques Contribution à la faisabilité d'un enseignement de la logique au lycée*. Thèse de l'université de Strasbourg.

Rogalski, J., & Rogalski, M. (2004). Contribution à l'étude des modes de traitement de la validité de l'implication par de futurs enseignants de mathématiques. In *Annales de Didactique et de Sciences Cognitives, Vol. 9* (pp. 175–203), Actes du colloque Argentoratum de juillet 2002.

Russell, B. (1903). *Les principes de la mathématique, traduction française, in RUSSEL, Ecrits de logique philosophique*. PUF: Paris 1989.

Selden, A., & Selden, J. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29*, 123–151.

Sierpinska, A., & Lerman, S. (1996). Epistemologies of mathematics and of mathematics education. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, C. Laborde (Eds.), *International handbook of mathematics education* (pp. 827–876). Dordrecht: Kluwer.

Sinaceur, H. (1991). *Corps et Modèles*. Paris: Vrin.

Sinaceur, H. (2001). Alfred Tarski, semantic shift, heuristic shift in metamathematics. *Synthese, 126*, 49–65.

Tarski, A. (1933a). The concept of truth in the language of deductive sciences. English translation in Tarski (1983), pp. 152–278.

Tarski, A. (1933b) Le concept de vérité dans les langages formalisés. French translation in Tarski (1972), pp. 157–269.

Tarski, A. (1936a). Introduction to logic and to the methodology of deductive sciences. French translation in Tarski (1969).

Tarski, A. (1936b). *Introduction to logic and to the methodology of deductive sciences* (*4th edition, 1994*). New York: Oxford University Press.

Tarski, A. (1936c). Sur le concept de conséquence. *Logique, Sémantique et Métamathématique, 1*, 141–152. Armand Colin, 1972.

Tarski, A. (1944a). The semantic conception of truth. *Philosophy and Phenomenological Research, 4*, 13–47.

Tarski, A. (1944). La conception sémantique de la vérité et les fondements de la sémantique. French translation in *Logique, sémantique et métamathématique, Vol. 2* (pp. 265–305). Paris: Armand Colin, 1972.

Tarski, A. (1969). *Introduction à la logique*. Paris-Louvain: Gauthier-Villard.

Tarski, A. (1972). *Logique, sémantique et métamathématique, Vol. 1*. Paris: Armand Colin.

Tarski, A. (1974). *Logique, sémantique et métamathématique, Vol. 2*. Paris: Armand Colin.

Tarski, A. (1983). *Logic, semantics, metamathematics, papers from 1923 to 1938*. Indianapolis: John Corcoran.

Wason, P. C., & Johnson-Laird, P.·N. (1977). A theoretical analysis of insight into a reasoning task. In P. N. Johnson-Laird, & P. C. Wason (Eds.), *Thinking: readings in cognitive science,* (pp. 143–157). Open University. Cited in Richard (1990).

Wittgenstein, L. (1921). *Tractatus logico-philosophicus*. Annalen der Naturphilosophie, Leipzig. French translation in Wittgenstein (1993); English translation in London: Routledge &Kegan Paul Ltd, 1922, 1961.

Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus*. C. K. Ogden Trans. London: Kegan Paul. D. F. Pears and B. F. McGuinnes, trans. London Routledge (1961).

Wittgenstein, L. (1993). *Tractatus logico-philosophicus*, traduction française G. G. Granger. Paris: Gallimard.